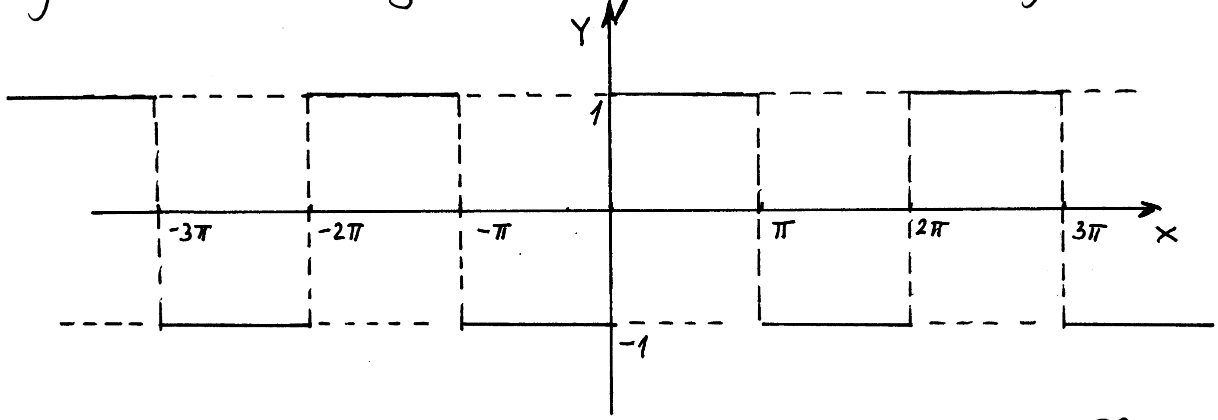
(#) F-ju definisanu grafikom pretvoriti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Rj. Primjetimo da je data f-ja periodična, perioda 2π, pa je možemo pretvoriti u Furijer-ov red. Kada je x-osa data u radijanima Furijer-ov red izgleda $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

gdje se Furijer-ovi koeficijenti računaju u obliku (za 2π per. f-ju)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{2}\int_{0}^{\pi} 1\,dx = \left(-\frac{1}{\pi}\right)\overbrace{x\Big|_{-\pi}^{0}}^{0-(-\pi)} + \frac{1}{\pi}x\Big|_0^{\pi} = -1+1 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \left(-\frac{1}{\pi}\right)\frac{1}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} b_n \sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \left(-\frac{1}{\pi}\right)\left(-\frac{1}{n}\right)\cos nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi}$$

$$= \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1) = \frac{2}{n\pi}(1-\cos n\pi)$$

$$1-\cos n\pi = 1-(-1)^n = \begin{cases} 0, & n=2k \\ 2, & n=2k+1 \end{cases} \quad k=0,1,2,\ldots$$

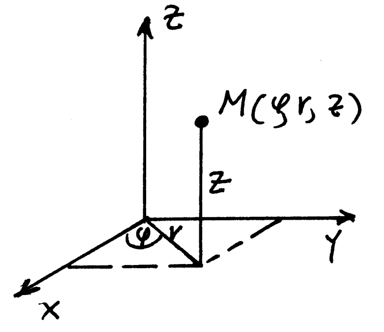
$$\boxed{\sin(2k+1)\frac{\pi}{2} = (-1)^k}$$

$$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$ traženi Furijerov red

$$f\left(\frac{\pi}{2}\right) = 1 \qquad \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\frac{\pi}{2}}{2k+1} = 1 \implies \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$ tražena suma

(#) Dat je trostruki integral $\int_0^{2\pi} d\varphi \int_0^{2} r^3 dr \int_0^{\sqrt{4-r^2}} dz$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

Rj: U cilindričnim koordinatama proizvoljna tačka M je opisana na sljedeći način



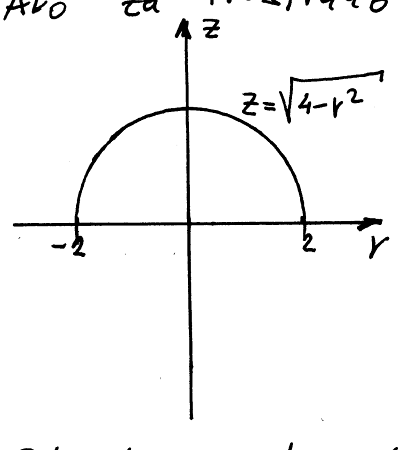
$$\Omega : \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \\ 0 \le z \sqrt{4-r^2} \end{cases}$$

Na osnovu izgleda oblasti Ω vidimo da je projekcija figure na xOy ravan oblika

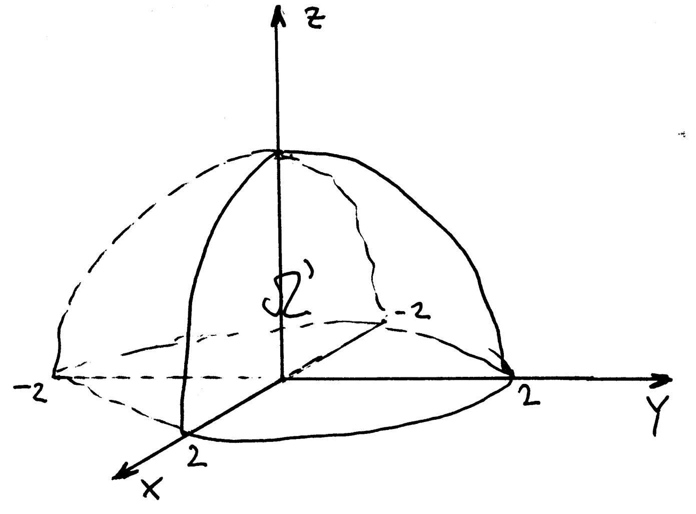
$$D : \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 2 \end{cases}$$

tj. krug sa centrom u koordinatnom početku poluprečnika 2.

Ako za fiksirano φ posmatramo rOz ravan imamo



Prema tome oblast integracije Ω je polulopta



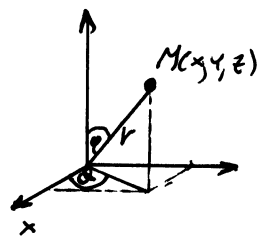
Cilindrične koordinate glase

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$dx\,dy = r\,dr\,d\varphi$

Tako da bi prelaskom na pravougaone koordinate sad imali

$$\int_0^{2\pi} d\varphi \int_0^{2} r^3\,dr \int_0^{\sqrt{1-r^2}} dz = \iiint_{\Omega} r^2\, r\,dr\,d\varphi\,dz = \left| \begin{array}{l} \text{prelazimo na pravougaone} \\ \text{koordinate} \\ \Omega \xrightarrow{\text{transformiše}} \Omega' \\ r\,dr\,d\varphi = dx\,dy \\ r^2 = r^2(\sin^2\varphi + \cos^2\varphi) \\ \quad = r^2\sin^2\varphi + r^2\cos^2\varphi \\ \quad = (r\sin\varphi)^2 + (r\cos\varphi)^2 \\ \quad = x^2 + y^2 \end{array} \right| =$$

$$= \iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz$$



Sferne koordinate glase

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$
$$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$\Omega' \xrightarrow{\text{transformiše}} \Omega'' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$

$$\iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| = \iiint_{\Omega''} r^2\sin^2\varphi\; r^2\sin\varphi\,dr\,d\theta\,d\varphi =$$

traženo rješenje

$$= \int_0^{2\pi} d\theta \int_0^{2} r^4\,dr \int_0^{\pi/2} \sin^3\varphi\,d\varphi \overset{(*)}{=} \theta\Big|_0^{2\pi} \cdot \frac{1}{5}r^5\Big|_0^{2} \cdot \frac{2}{3} = \frac{2^7}{15}\pi$$

$$\int_0^{\pi/2} \sin^3\varphi\,d\varphi = \int_0^{\pi/2} \sin\varphi\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\,d\varphi = \left| \begin{array}{l} d(\sin\varphi) = \cos\varphi\,d\varphi \\ d(\cos\varphi) = -\sin\varphi\,d\varphi \end{array} \right| = -\int_0^{\pi/2} (1-\cos^2\varphi)\,d(\cos\varphi)$$

$$= -\left(\cos\varphi\Big|_0^{\frac{\pi}{2}} - \frac{1}{3}\cos^3\varphi\Big|_0^{\frac{\pi}{2}}\right) = -\left((0-1) - \frac{1}{3}(0-1)\right) = -\left(-1+\frac{1}{3}\right) = \frac{2}{3} \quad \ldots(*)$$

(#) Izračunati krivoliniski integral $\int_L (x-y)\,ds$ po kružnoj liniji $x^2+y^2=ax$.

Rj:

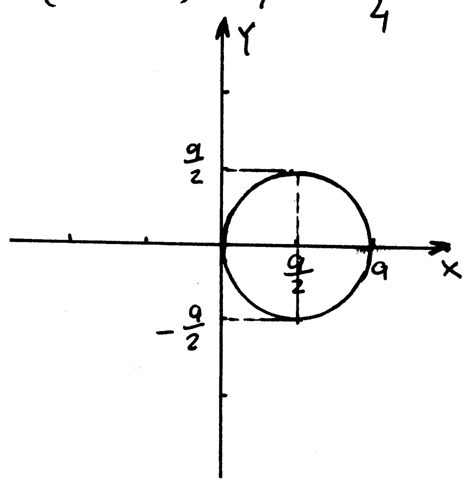
$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

krug sa centrom u $C(\frac{a}{2},0)$ poluprečnika $r=\frac{a}{2}$



Kako se računa krivoliniski integral $\int_L f(x,y)\,ds$?

Ako je kriva L data u obliku f-je $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_L f(x,y)\,ds=\int_a^b f(x,\eta(x))\sqrt{1+(y')^2}\,dx.$$

Ako je kriva L data u parametarskom obliku

$$\begin{cases} x=\mu(t)\\ y=\theta(t)\\ t_1\le t\le t_2\end{cases}$$

tada $\int_L f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{\mu'(t)^2+\theta'(t)^2}\,dt$

Prisjetimo se polarnih koordinata $x=r\cos\varphi$, $y=r\sin\varphi$.

Ako pomjerimo centar u x-osi za $\frac{a}{2}$ i fiksiramo r na $\frac{a}{2}$ imamo da je

$$L:\begin{cases} x=\frac{a}{2}+\frac{a}{2}\cos\varphi\\ y=\frac{a}{2}\sin\varphi\\ 0\le\varphi\le 2\pi\end{cases}$$

$$x'=-\frac{a}{2}\sin\varphi$$

$$y'=\frac{a}{2}\cos\varphi$$

$$(x')^2+(y')^2=$$

$$=\frac{a^2}{4}\sin^2\varphi+\frac{a^2}{4}\cos^2\varphi$$

$$=\frac{a^2}{4}$$

$$\sqrt{x'^2+y'^2}=\frac{a}{2}$$

$$\int_L (x-y)\,ds=\int_0^{2\pi}\left(\frac{a}{2}+\frac{a}{2}\cos\varphi-\frac{a}{2}\sin\varphi\right)\cdot\frac{a}{2}\,d\varphi=\int_0^{2\pi}\left(\frac{a^2}{4}+\frac{a^2}{4}\cos\varphi-\frac{a^2}{4}\sin\varphi\right)d\varphi=$$

$$=\frac{a^2}{4}\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\sin\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\cos\varphi\Big|_0^{2\pi}=\frac{a^2}{4}\cdot 2\pi=\frac{a^2\pi}{2}$$

traženo rješenje

(#) Izračunati površinski integral $\iint\limits_{(S)} \sqrt{-x^2+4}\, dS$, gdje je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$.

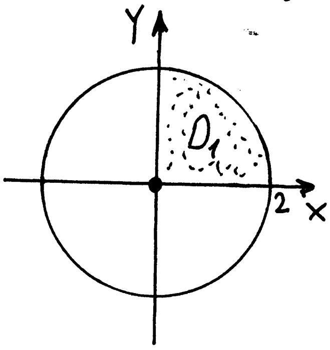
Rj. Skicirajmo površ $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$

u xOy ravni

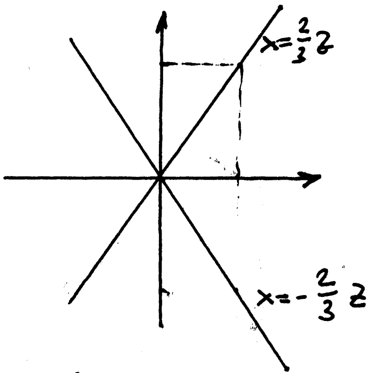
$$\frac{x^2}{4}+\frac{y^2}{4}=0$$

za $z=0$, $x^2+y^2=0$

tačka $(0,0)$



u xOz ravni



$$\frac{x^2}{4}=\frac{z^2}{9}$$

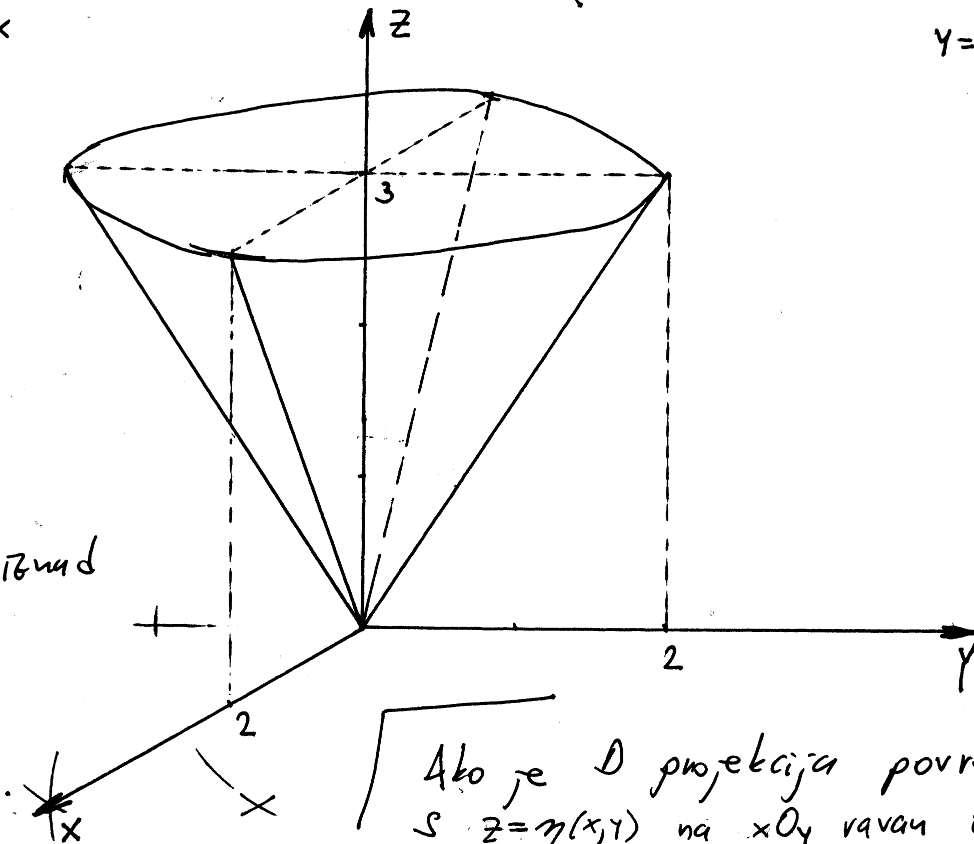
$$x^2=\frac{4}{9}z^2$$

$$x=\pm\frac{2}{3}z$$

yOz ravan

$$y=\pm\frac{2}{3}z$$

za $z=3$ $x^2+y^2=4$



$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$$

$$z^2=\frac{9}{4}(x^2+y^2)$$

Kako je data površ iznad xOy ravni

$$z=\frac{3}{2}\sqrt{x^2+y^2}$$

$$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{3x}{2\sqrt{x^2+y^2}} \qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint\limits_{S} f(x,y,z)\,dS=\iint\limits_{D} f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)}=\frac{13x^2+13y^2}{4(x^2+y^2)}=\frac{13}{4}$$

Primjetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = \frac{\sqrt{13}}{2} \iint\limits_{D} \sqrt{-x^2+4}\, dx\,dy = 4 \cdot \frac{\sqrt{13}}{2} \iint\limits_{D_1} \sqrt{4-x^2}\, dx\,dy$$

gdje je $D_1 : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = 2\sqrt{13} \int_0^2 \sqrt{4-x^2}\, dx \int_0^{\sqrt{4-x^2}} dy = 2\sqrt{13} \int_0^2 (4-x^2)\, dx =$$

$$= 2\sqrt{13}\left(4x\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2\right) = 2\sqrt{13}\left(8 - \frac{8}{3}\right) = 2\sqrt{13} \cdot \frac{\overset{16}{24-8}}{3}$$

$$= \frac{32}{3}\sqrt{13}$$ traženo rješenje